

Mixed stochastic differential equations: Existence and uniqueness result

José Luís da Silva

CCM, University of Madeira, Campus da Penteada,
9020-105 Funchal, Portugal.

Email: luis@uma.pt

Mohamed Erraoui

Université Cadi Ayyad, Faculté des Sciences Semlalia,
Département de Mathématiques, B.P. 2390, Marrakech, Maroc

Email: erraoui@uca.ma

El Hassan Essaky

Université Cadi Ayyad, Faculté Poly-disciplinaire
Laboratoire de Modélisation et Combinatoire
Département de Mathématiques et d'Informatique B.P. 4162,
Safi, Maroc.

Email: essaky@uca.ma

Abstract

In this paper we shall establish an existence and uniqueness result for solutions of multidimensional, time dependent, stochastic differential equations driven simultaneously by a multidimensional fractional Brownian motion with Hurst parameter $H > 1/2$ and a multidimensional standard Brownian motion under a weaker condition than the Lipschitz one.

Keywords: Fractional Brownian motion, stochastic differential equations, weak and strong solution, Bihari's type lemma.

1 Introduction

The fractional Brownian motion (fBm for short) $B^H = \{B^H(t), t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian self-similar process with stationary increments. This process was introduced by Kolmogorov [10] and studied by Mandelbrot and Van Ness in [13], where a stochastic integral representation in terms of a standard Brownian motion (Bm for short) was established. The parameter H is called Hurst index from the statistical analysis, developed

by the climatologist Hurst [7]. The self-similarity and stationary increments properties make the fBm an appropriate model for many applications in diverse fields from biology to finance. From the properties of the fBm it follows that, for every $\alpha > 0$

$$\mathbb{E}(|B^H(t) - B^H(s)|^\alpha) = \mathbb{E}(|B^H(1)|^\alpha) |t - s|^{\alpha H}.$$

As a consequence of the Kolmogorov continuity theorem, we deduce that there exists a version of the fBm B^H which is a continuous process and whose paths are γ -Hölder continuous for every $\gamma < H$. Therefore, the fBm with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale and then the Itô approach to the construction of stochastic integrals with respect to fBm is not valid. Two main approaches have been used in the literature to define stochastic integrals with respect to fBm with Hurst parameter H . Pathwise Riemann-Stieltjes stochastic integrals can be defined using Young's integral [18] in the case $H > \frac{1}{2}$. When $H \in (\frac{1}{4}, \frac{1}{2})$, the rough path analysis introduced by Lyons [12] is a suitable method to construct pathwise stochastic integrals.

A second approach to develop a stochastic calculus with respect to the fBm is based on the techniques of Malliavin calculus. The divergence operator, which is the adjoint of the derivative operator, can be regarded as a stochastic integral, which coincides with the limit of Riemann sums constructed using the Wick product. This idea has been developed by Decreusefond and Üstünel [6], Carmona, Coutin and Montseny [5], Alòs, Mazet and Nualart [1, 2], Alòs and Nualart [3] and Hu [17], among others. The integral constructed by this method has zero mean.

Let $T > 0$ be a fixed time and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a given filtered complete probability space with $(\mathcal{F}_t)_{t \in [0, T]}$ being a filtration that satisfies the usual hypotheses. The aim of this paper is to study the following stochastic differential equation (SDE for short) on \mathbb{R}^n

$$X(t) = x_0 + \int_0^t b(s, X(s)) ds + \int_0^t \sigma_W(s, X(s)) dW(s) + \int_0^t \sigma_H(s, X(s)) dB^H(s), \quad (1.1)$$

where $t \in [0, T]$, $x_0 \in \mathbb{R}^n$, W is a m -dimensional standard \mathcal{F}_t -Bm and B^H a d -dimensional \mathcal{F}_t -adapted fBm. The main difficulty when considering Equation (1.1) lies in the fact that both stochastic integrals are dealt in different ways. However, the integral with respect to the Bm is an Itô integral, while the integral with respect to the fBm has to be understood in the pathwise sense. Mixing the two integrals makes things difficult, forcing to consider very smooth coefficients to prove existence and uniqueness of solution to Equation (1.1).

It is well known that, under suitable assumptions on the coefficients b, σ_W, σ_H (see below), the Equation (1.1) has a unique solution which is $(H - \varepsilon)$ -Hölder continuous, for all $\varepsilon > 0$. This result was first considered in [11], where unique solvability was proved for time-independent coefficients and zero drift. Later, in [20], existence of solution to (1.1) was proved under less restrictive assumptions, but only locally, i.e. up to a random time. In [8], global existence and uniqueness of solution to the Equation (1.1) was established under the assumption that W and B^H are independent. The latter result was obtained in [14, 15] without the independence assumption. We stress on the fact that all these works consider the Lipschitz case. It should be noted, in addition, that the Lipschitz condition is the most used to establish the pathwise uniqueness for ordinary and SDEs via the Gronwall lemma. Thus, the following question appears naturally: are there any weaker conditions than the Lipschitz continuity under which the SDE (1.1) has a unique strong solution?

In order to answer the above question our approach is to prove that the Euler's polygonal approximations converge uniformly in $t \in [0, T]$, in probability, to a process, which we show to be the strong solution. The basic tools are the pathwise uniqueness for the SDE (1.1), tightness of the sequence of the laws of Euler's approximations and the Skorokhod's embedding theorem. It is important to note that the linear growth condition and the continuity of the coefficients are sufficient for the convergence of the Stieltjes and Itô integrals. However, the integral with respect the fBm needs more regularity. To prove the convergence in probability we use an elementary result due to Gyongy and Krylov [9] which highlights the famous result of Yamada and Watanabe saying that pathwise uniqueness implies uniqueness in law. It is worth mentioning that the pathwise uniqueness property for the SDE (1.1) is obtained under weak assumption than the Lipschitz condition. More precisely our conditions are based on the modulus of continuity of the coefficients that achieve pathwise uniqueness using Bihari's type lemma. It should be noted that such conditions are considered by many authors for the existence and uniqueness of solutions of different kind of equations where the Bihari's lemma is the cornerstone in the proof of these results.

The article is organized as follows. In Section 2, we state our assumptions on the coefficients b , σ_W and σ_H of Equation (1.1), recall briefly the deterministic fractional calculus in order to define the integral with respect to fBm and introduce proper normed spaces. In addition, we give the definition of strong, weak solution and pathwise uniqueness of Equation (1.1). In Section 3, the pathwise uniqueness property for the solutions of Equation (1.1) is proved (see Theorem 7 below). Finally, in Section 4, we define the Euler approximations sequence and prove that it is tight. Moreover, we show that these approximations converge in probability to a process which turns out to be a strong solution of the SDE (1.1), cf. Theorem 9 below. In the Appendix, we recall some technical results which play a great role in this work. We also show a version of Bihari's lemma which will be used in the proof of pathwise uniqueness to SDE (1.1).

2 Preliminaries

Throughout this paper we assume that the coefficients b , σ_W and σ_H , which are continuous, satisfy, for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$, the following hypotheses **(H.1)** and **(H.2)**:

Hypothesis (H.1). The functions b and σ_W have a linear growth and satisfy suitable modulus of continuity with respect to the variable x uniformly in t .

Hypothesis **(H.1)** means that b and σ_W satisfy

$$\textbf{(H.1.1)} \quad |b(t, x)| \leq K(1 + |x|),$$

$$\textbf{(H.1.2)} \quad |b(t, x) - b(t, y)|^2 \leq \varrho(|x - y|^2)$$

$$\textbf{(H.1.3)} \quad |\sigma_W(t, x)| \leq K(1 + |x|),$$

$$\textbf{(H.1.4)} \quad |\sigma_W(t, x) - \sigma_W(t, y)|^2 \leq \varrho(|x - y|^2),$$

where ϱ is a concave increasing function from \mathbb{R}_+ to \mathbb{R}_+ such that $\varrho(0) = 0$, $\varrho(u) > 0$ for $u > 0$ and for some $q > 1$ we have

$$\int_{0+} \frac{du}{\varrho^q(u^{1/q})} = \infty. \quad (2.1)$$

Hypothesis (H.2). The function σ_H is continuously differentiable in the second variable x . Its derivative, with respect to x , is bounded, Lipschitz with respect to the same variable uniformly with respect to the first variable t . Moreover, both σ_H and its derivative are β -Hölder with respect to the first variable t uniformly with respect to the second variable.

Hypothesis (H.2) means that σ_H and its derivative satisfy

$$(H.2.1) \quad |\partial_{x_i} \sigma_H(t, x)| \leq K$$

$$(H.2.2) \quad |\partial_{x_i} \sigma_H(t, x) - \partial_{x_i} \sigma_H(t, y)| \leq K |x - y|$$

$$(H.2.3) \quad |\sigma_H(t, x) - \sigma_H(s, x)| + |\partial_{x_i} \sigma_H(t, x) - \partial_{x_i} \sigma_H(s, x)| \leq K |s - t|^\beta.$$

Example 1. Let us give two examples of such function ϱ . Let $q > 1$ and δ be sufficiently small. Define

$$\varrho_1(u) := \begin{cases} u \log^{1/q}(u^{-1}), & 0 \leq u \leq \delta \\ \delta \log^{1/q}(\delta^{-1}) + \varrho'_1(\delta_-)(u - \delta), & u > \delta. \end{cases}$$

$$\varrho_2(u) := \begin{cases} u \log^{1/q}(u^{-1}) \log^{1/q}(\log(u^{-1})), & 0 \leq u \leq \delta \\ \delta \log^{1/q}(\delta^{-1}) \log^{1/q}(\log(\delta^{-1})) + \varrho'_2(\delta_-)(u - \delta), & u > \delta. \end{cases}$$

It is easy to see that, for $i = 1, 2$, the function ϱ_i is concave nondecreasing function satisfying (2.1).

We begin by a brief review of the deterministic fractional calculus. We start with the definition of the integral with respect to fBm as a generalized Lebesgue-Stieltjes integral, following the work of Zähle [20]. We fix $\alpha \in (0, 1)$. The Weyl-Marchaud derivatives of $f : [a, b] \rightarrow \mathbb{R}^n$ are given by:

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbb{1}_{(a,b)}(x)$$

and

$$D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbb{1}_{(a,b)}(x),$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Gamma function. Assuming that $D_{a+}^\alpha f_{a+} \in L^1[a, b]$ and $D_{b-}^{1-\alpha} g_{b-} \in L^\infty[a, b]$, where $g_{b-}(x) = g(x) - g(b-)$, the generalized (fractional) Lebesgue-Stieltjes integral of f with respect to g is defined as

$$\int_a^b f dg := (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx. \quad (2.2)$$

If $a \leq c < d \leq b$ then we have

$$\int_c^d f dg = \int_a^b \mathbb{1}_{(c,d)} f dg.$$

It follows from the Hölder continuity of B^H that $D_{b-}^{1-\alpha} B_{b-}^H \in L^\infty[a, b]$ almost surely (a.s. for short). Then, for a function f with $D_{a+}^\alpha f \in L^1[a, b]$, we can define the integral with respect to B^H through (2.2).

Let $0 < \alpha < 1/2$ and $\mu \in (0, 1]$. We will consider the following normed spaces:

1. C^μ is the space of μ -Hölder continuous functions $f : [0, T] \rightarrow \mathbb{R}^d$, equipped with the norm

$$\|f\|_\mu := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\mu} < \infty,$$

where

$$\|f\|_\infty := \sup_{0 \leq t \leq T} |f(t)|.$$

2. C_0^μ denotes the space of μ -Hölder continuous functions $f : [0, T] \rightarrow \mathbb{R}^d$ such that

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{0 < |t-s| < \varepsilon} \frac{|f(t) - f(s)|}{(t - s)^\mu} \right) = 0.$$

We note that C_0^μ is complete and separable with respect to the norm $\|\cdot\|_\mu$.

3. $W_0^{\alpha, \infty}$ is the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha, \infty} := \sup_{0 \leq t \leq T} \|f\|_{\alpha, t} < \infty,$$

where

$$\|f\|_{\alpha, t} := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t - s)^{\alpha+1}} ds.$$

4. Finally, $W_T^{1-\alpha, \infty}$ denotes the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}^m$ such that

$$\|f\|_{1-\alpha, \infty, T} := \sup_{0 \leq t \leq T} \|f\|_{1-\alpha, \infty, t} < \infty,$$

where

$$\|f\|_{1-\alpha, \infty, t} := \sup_{0 \leq u < v < t} \left(\frac{|f(v) - f(u)|}{(v - u)^{1-\alpha}} + \int_u^v \frac{|f(y) - f(u)|}{(y - u)^{2-\alpha}} dy \right).$$

Hence, it is clear that

$$\sup_{0 \leq u < v < t} |D_{v-}^{1-\alpha} B_{v-}^H(u)| \leq \frac{1}{\Gamma(\alpha)} \|B^H\|_{1-\alpha, \infty, t} < \infty,$$

where the last inequality is a consequence of that fact that the random variable $\|B^H\|_{1-\alpha, \infty, t}$ has moments of all orders, see Lemma 7.5 in Nualart and Rascanu [16]. Thus, the stochastic integral with respect to the fBm admits the following estimate

$$\left| \int_0^t f(s) dB^H(s) \right| \leq \frac{1}{\Gamma(\alpha)} \|B^H\|_{1-\alpha, \infty, t} \|f\|_{\alpha, 1, t}, \quad (2.3)$$

where

$$\|f\|_{\alpha,1,t} := \int_0^t \frac{|f(s)|}{s^\alpha} ds + \int_0^t \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy ds.$$

We give the definition of strong and weak solution as well as pathwise uniqueness for Equation (1.1).

Definition 2 (Strong solution). By a strong solution of Equation (1.1) we mean an \mathcal{F}_t -adapted continuous process $X(t), t \in [0, T]$ such that there exists an increasing sequence of stopping times $(T_R)_{R>0}$ satisfying $\lim_{R \rightarrow \infty} T_R = T$ a.s. and for any $R > 0$, we have

$$1. \sup_{t \in [0, T]} \mathbb{E} [\|X(t \wedge T_R)\|_{\alpha, t}^2] < \infty.$$

2. The equation

$$\begin{aligned} X(t \wedge T_R) &= x_0 + \int_0^{t \wedge T_R} b(s, X(s)) ds + \int_0^{t \wedge T_R} \sigma_W(s, X(s)) dW(s) \\ &\quad + \int_0^{t \wedge T_R} \sigma_H(s, X(s)) dB^H(s), \end{aligned} \tag{2.4}$$

holds a.s..

Definition 3 (Weak solution). By a weak solution of Equation (1.1) we mean a triplet (X, W, B^H) , (Ω, \mathcal{F}, P) and $(\mathcal{F}_t)_{t \in [0, T]}$, such that

1. (Ω, \mathcal{F}, P) is a probability space, and $(\mathcal{F}_t)_{t \in [0, T]}$ is a filtration, of sub- σ -algebra of \mathcal{F} , satisfying the usual conditions.
2. $W = (W_t, \mathcal{F}_t)_{t \in [0, T]}$ is a Bm, $B^H = (B_t^H)_{t \in [0, T]}$ is a fBm and $X = (X_t, \mathcal{F}_t)_{t \in [0, T]}$ is a continuous and \mathcal{F}_t -adapted process satisfying a.s. the Equation (2.4) for some increasing sequence of stopping times $(T_R)_{R>0}$ such that $\lim_{R \rightarrow \infty} T_R = T$ a.s..

Definition 4 (Pathwise uniqueness). We say that pathwise uniqueness holds for Equation (1.1) if, whenever (X, W, B^H) and (\tilde{X}, W, B^H) are two weak solutions of Equation (1.1) defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ then X and \tilde{X} are indistinguishable.

3 Pathwise uniqueness

In this section we investigate the pathwise uniqueness of a solution for Equation (1.1), cf. Theorem 7 below, where we make use of the so-called Bihari's type lemma (see Lemma 14 in Appendix).

Let X be a solution of Equation (1.1). For $R > 0$, we define the following stopping time

$$T_R := \inf \left\{ t \geq 0, \|B^H\|_{1-\alpha, \infty, t} \geq R \right\} \wedge T,$$

For every positive constant R , we define the stochastic processes X_R by

$$X_R(t) := X(t \wedge T_R), \quad t \in [0, T].$$

Then it is easy to see that the following equation

$$\begin{aligned} X_R(t) &= x_0 + \int_0^{t \wedge T_R} b(s, X(s)) ds + \int_0^{t \wedge T_R} \sigma_W(s, X(s)) dW(s) \\ &\quad + \int_0^{t \wedge T_R} \sigma_H(s, X(s)) dB^H(s) \end{aligned}$$

holds almost surely. We have the following Lemma.

Lemma 5. *For any integer $N \geq 1$ and $R > 0$, there exists a positive constant C_N such that*

$$\sup_{t \in [0, T]} \mathbb{E} [\|X_R\|_{\alpha, t}^{2N}] \leq C_N R^{2N}.$$

Proof. Along the proof C_N will denote a generic positive constant, which may vary from line to line and may depend on N and other parameters of the problem. It follows from the convexity of x^{2N} that

$$\begin{aligned} \mathbb{E} [\|X_R\|_{\alpha, t}^{2N}] &\leq C_N \left\{ |x_0|^{2N} + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} b(s, X(s)) ds \right\|_{\alpha, t}^{2N} \right] \right. \\ &\quad + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} \sigma_W(s, X(s)) dW(s) \right\|_{\alpha, t}^{2N} \right] \\ &\quad \left. + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} \sigma_H(s, X(s)) dB^H(s) \right\|_{\alpha, t}^{2N} \right] \right\} \\ &= C_N (|x_0|^{2N} + A_1 + A_2 + A_3). \end{aligned}$$

Furthermore we have

$$\begin{aligned} &\left\| \int_0^{\cdot \wedge T_R} b(s, X(s)) ds \right\|_{\alpha, t} \\ &\leq \int_0^{t \wedge T_R} |b(s, X(s))| ds + \int_0^t (t-s)^{-\alpha-1} \int_{s \wedge T_R}^{t \wedge T_R} |b(u, X(u))| du ds \\ &\leq \int_0^t |b(s \wedge T_R, X(s \wedge T_R))| ds \\ &\quad + \int_0^t (t-s)^{-\alpha-1} \int_s^t |b(u \wedge T_R, X(u \wedge T_R))| du ds \\ &\leq \int_0^t |b(s \wedge T_R, X(s \wedge T_R))| ds \\ &\quad + \frac{1}{\alpha} \int_0^t (t-r)^{-\alpha} |b(r \wedge T_R, X(r \wedge T_R))| dr \\ &\leq C_{\alpha, T} \int_0^t (t-r)^{-\alpha} |b(r \wedge T_R, X(r \wedge T_R))| dr \end{aligned}$$

where $C_{\alpha,T}$ is a constant depending on α and T . Using the linear growth assumption in **(H.1.1)**, Hölder's inequality and the fact that $\alpha < \frac{1}{2}$, we obtain

$$\begin{aligned} A_1 &\leq C_N \mathbb{E} \left[\left(1 + \int_0^t \frac{|X_R(s)|}{(t-s)^\alpha} ds \right)^{2N} \right] \\ &\leq C_N \mathbb{E} \left[\left(1 + \int_0^t |X_R(s)|^2 ds \right)^N \right] \\ &\leq C_N \left(1 + \int_0^t \mathbb{E} [|X_R(s)|^{2N}] ds \right). \end{aligned}$$

We have also that

$$\begin{aligned} A_2 &\leq C_N \mathbb{E} \left[\left| \int_0^{t \wedge T_R} \sigma_W(s, X(s)) dW(s) \right|^{2N} \right] \\ &\quad + C_N \mathbb{E} \left[\left(\int_0^t (t-s)^{-\alpha-1} \left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(u, X(u)) dW(u) \right| ds \right)^{2N} \right] \\ &= A_{21} + A_{22}. \end{aligned}$$

For A_{21} , using the linear growth assumption in **(H.1.3)**, the Burkholder and Hölder inequalities, we obtain

$$\begin{aligned} A_{21} &\leq C_N \mathbb{E} \left[\int_0^{t \wedge T_R} |\sigma_W(s, X(s))|^{2N} ds \right] \\ &\leq C_N \mathbb{E} \left[\int_0^t |\sigma_W(s \wedge T_R, X(s \wedge T_R))|^{2N} ds \right] \\ &\leq C_N \left(1 + \int_0^t \mathbb{E} [|X_R(s)|^{2N}] ds \right) \end{aligned}$$

For A_{22} , again the Burkholder and Hölder inequalities give

$$\begin{aligned} A_{22} &\leq C_N \left(\int_0^t \frac{ds}{(t-s)^{\alpha+\frac{1}{2}}} \right)^{2N-1} \int_0^t (t-s)^{-\alpha-\frac{1}{2}-N} \mathbb{E} \left[\left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(u, X(u)) dW(u) \right|^{2N} \right] ds \\ &\leq C_N \int_0^t (t-s)^{-\alpha-\frac{3}{2}} \mathbb{E} \left[\int_{s \wedge T_R}^{t \wedge T_R} |\sigma_W(u, X(u))|^{2N} du \right] ds \\ &\leq C_N \int_0^t (t-s)^{-\alpha-\frac{3}{2}} \mathbb{E} \left[\int_s^t |\sigma_W(u \wedge T_R, X(u \wedge T_R))|^{2N} du \right] ds. \end{aligned}$$

Applying now Fubini's theorem and using the growth assumption in **(H.1.3)**, we obtain

$$A_{22} \leq C_N \left(\int_0^t (t-s)^{-\alpha-\frac{1}{2}} \left(1 + \mathbb{E} [|X_R(s)|^{2N}] \right) ds \right).$$

Thus

$$A_2 \leq C_N \left(1 + \int_0^t (t-s)^{-\alpha-\frac{1}{2}} \mathbb{E} \left[|X_R(s)|^{2N} \right] ds \right).$$

Let us remark that, for $t \in [0, T]$, we have

$$\int_0^{t \wedge T_R} \sigma_H(s, X(s)) dB^H(s) = \int_0^t \sigma_H(s \wedge T_R, X(s \wedge T_R)) dB^H(s \wedge T_R). \quad (3.1)$$

Then it follows from Proposition 12 (jj), in the Appendix, that

$$A_3 \leq C_N R^{2N} \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \mathbb{E} [\|X_R\|_{\alpha,s}^{2N}]) ds.$$

Putting all the estimates obtained for A_1 , A_2 and A_3 together, we obtain

$$\mathbb{E} [\|X_R\|_{\alpha,t}^{2N}] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(t,s) \mathbb{E} [\|X_R\|_{\alpha,s}^{2N}] ds, \quad (3.2)$$

where

$$\varphi(t,s) := s^{-\alpha} + (t-s)^{-\alpha-1/2}.$$

Therefore, since the right hand side of Equation (3.2) is an increasing function of t , we have

$$\sup_{0 \leq s \leq t} \mathbb{E} [\|X_R\|_s^{2N}] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(s,t) \sup_{0 \leq u \leq s} \mathbb{E} [\|X_R\|_{\alpha,u}^{2N}] ds.$$

As a consequence, by the Gronwall type lemma (Lemma 7.6 in [16]), we deduce the desired estimate. \square

Let X and Y be two solutions of Equation (1.1) defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t \in [0,T]}), P)$. For $M > 0$, we define the following stopping time

$$\tau_M := \inf \{ t : \|X\|_{\alpha,t} \vee \|Y\|_{\alpha,t} > M \} \wedge T.$$

Now for every positive constants R and M , we define the stochastic processes $X_{R,M}$ (resp. $Y_{R,M}$) by

$$X_{R,M}(t) := X(t \wedge T_R \wedge \tau_M), \quad t \in [0, T],$$

(resp. $Y_{R,M}(t) := Y(t \wedge T_R \wedge \tau_M)$, $t \in [0, T]$).

Lemma 6. *Under Hypotheses (H.1) and (H.2), there exists a positive constant $C_{R,M}$ such that for $t \in [0, T]$,*

$$\begin{aligned} & \mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2] \\ & \leq C_{R,M} \int_0^t \varphi(s,t) \left[\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2] + \varrho \left(\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2] \right) \right] ds. \end{aligned} \quad (3.3)$$

Proof. The proof of this result is long and technical. It is divided into several parts. First we have

$$\begin{aligned}
X_{R,M}(t) - Y_{R,M}(t) &= \int_0^{t \wedge T_R \wedge \tau_M} (b(s, X(s)) - b(s, Y(s))) ds \\
&\quad + \int_0^{t \wedge T_R \wedge \tau_M} (\sigma_W(s, X(s)) - \sigma_W(s, Y(s))) dW(s) \\
&\quad + \int_0^{t \wedge T_R \wedge \tau_M} (\sigma_H(s, X(s)) - \sigma_H(s, Y(s))) dB^H(s) \\
&= B_1(t \wedge T_R \wedge \tau_M) + B_2(t \wedge T_R \wedge \tau_M) + B_3(t \wedge T_R \wedge \tau_M).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2 \\
&\leq 3 \left(\|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 + \|B_2(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 + \|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \right).
\end{aligned}$$

We have to estimated $\|B_i(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2$, $i \in \{1, 2, 3\}$. For the sake of conciseness, we define

$$\Delta(f)(s) = f(s, X(s)) - f(s, Y(s)), \quad f \in \{b, \sigma_W, \sigma_H\}.$$

Step 1: B_1 . Using simple estimations it is easy to see that

$$\begin{aligned}
\|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t} &\leq \int_0^{t \wedge T_R \wedge \tau_M} |\Delta(b)(s)| ds \\
&\quad + \int_0^t (t-s)^{-\alpha-1} \int_{s \wedge T_R \wedge \tau_M}^{t \wedge T_R \wedge \tau_M} |\Delta(b)(u)| du ds \\
&\leq \int_0^t |\Delta(b)(s \wedge T_R \wedge \tau_M)| ds \\
&\quad + \int_0^t (t-s)^{-\alpha-1} \int_s^t |\Delta(b)(u \wedge T_R \wedge \tau_M)| du ds \\
&\leq C_{\alpha,T} \int_0^t (t-r)^{-\alpha} |\Delta(b)(r \wedge T_R \wedge \tau_M)| dr.
\end{aligned}$$

We use the fact that $\alpha < \frac{1}{2}$, Hölder inequality and hypothesis **(H.1.2)** to obtain

$$\begin{aligned}
\|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 &\leq C_{\alpha,T}^2 \int_0^t \frac{|\Delta(b)(s \wedge T_R \wedge \tau_M)|^2}{(t-s)^\alpha} ds \\
&\leq C_{\alpha,T}^2 \int_0^t \frac{\varrho(|X_{R,M}(s) - Y_{R,M}(s)|^2)}{(t-s)^\alpha} ds \\
&\leq C_{\alpha,T}^2 \int_0^t \varphi(s, t) \varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2) ds.
\end{aligned}$$

Step 2: B_3 . If $1 - H < \alpha < \min(\beta, 1/2)$, we have from Proposition 4.3 in [16] (see Proposition 11 (ii) in the Appendix) that

$$\|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq CR^2 \left(\int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \|\Delta(\sigma_H)(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,s} ds \right)^2.$$

Now using the assumptions **(H.2)** and Lemma 7.1 in Nualart Rascanu [16] we obtain

$$\begin{aligned} & |\sigma_H(t, x_1) - \sigma_H(s, x_2) - \sigma_H(t, y_1) + \sigma_H(s, y_2)| \\ & \leq K |x_1 - x_2 - y_1 + y_2| + K |x_1 - y_1| |t - s|^\beta \\ & \quad + K |x_1 - y_1| (|x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Therefore

$$\begin{aligned} & \left| \sigma_H(t \wedge T_R \wedge \tau_M, X_{R,M}(t)) - \sigma_H(s \wedge T_R \wedge \tau_M, X_{R,M}(s)) \right. \\ & \quad \left. - \sigma_H(t \wedge T_R \wedge \tau_M, Y_{R,M}(t)) + \sigma_H(s \wedge T_R \wedge \tau_M, Y_{R,M}(s)) \right| \\ & \leq K \left[|X_{R,M}(t) - X_{R,M}(s) - Y_{R,M}(t) + Y_{R,M}(s)| + K |X_{R,M}(t) - Y_{R,M}(t)| |t - s|^\beta \right. \\ & \quad \left. + |X_{R,M}(t) - Y_{R,M}(t)| (|X_{R,M}(t) - X_{R,M}(s)| + |Y_{R,M}(t) - Y_{R,M}(s)|) \right]. \end{aligned}$$

Thus we have

$$\begin{aligned} & \|\Delta(\sigma_H)(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t} \\ & \leq K \left[|X_{R,M}(t) - Y_{R,M}(t)| + \int_0^t \frac{|X_{R,M}(t) - X_{R,M}(s) - Y_{R,M}(t) + Y_{R,M}(s)|}{(t-s)^{\alpha+1}} ds \right. \\ & \quad + |X_{R,M}(t) - Y_{R,M}(t)| \left(\int_0^t \frac{ds}{(t-s)^{\alpha-\beta+1}} + \int_0^t \frac{|X_{R,M}(t) - X_{R,M}(s)|}{(t-s)^{\alpha+1}} ds \right. \\ & \quad \left. \left. + \int_0^t \frac{|Y_{R,M}(t) - Y_{R,M}(s)|}{(t-s)^{\alpha+1}} ds \right) \right]. \end{aligned}$$

Now it is easy to see that

$$\begin{aligned} & \|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \\ & \leq CR^2 \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \|X_{R,M}\|_{\alpha,s}^2 + \|Y_{R,M}\|_{\alpha,s}^2) \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 ds \\ & \leq CR^2 M^2 \int_0^t \varphi(s, t) \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 ds. \end{aligned}$$

Spet 3: B_2 . Till now we have made estimates for pathwise integrals. As B_2 is a stochastic integral we need to use martingale type inequality. First we have

$$\|B_2(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq 2 \left(|B_2(t \wedge T_R \wedge \tau_M)|^2 + (\tilde{B}_2(t))^2 \right),$$

where

$$\tilde{B}_2(t) := \int_0^t \frac{\left| \int_{s \wedge T_R \wedge \tau_M}^{t \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right|}{(t-s)^{\alpha+1}} ds.$$

It then follows from Burkholder inequality and assumption **(H.1.4)** that

$$\begin{aligned} \mathbb{E}(|B_2(t \wedge T_R \wedge \tau_M)|^2) &\leq \mathbb{E} \left(\int_0^t |\Delta(\sigma_W)(s \wedge T_R \wedge \tau_M)|^2 ds \right) \\ &\leq C \mathbb{E} \left(\int_0^t \varrho(|X_{R,M}(s) - Y_{R,M}(s)|^2) ds \right) \\ &\leq C \mathbb{E} \left(\int_0^t \varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2) ds \right). \end{aligned}$$

\tilde{B}_2 : Using Hölder's inequality and Fubini's theorem we have

$$\begin{aligned} \mathbb{E} [|\tilde{B}_2(t)|^2] &\leq C \mathbb{E} \left[\int_0^t (t-s)^{-\frac{3}{2}-\alpha} \left| \int_{s \wedge T_R \wedge \tau_M}^{t \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right|^2 ds \right] \\ &\leq C \int_0^t (t-s)^{-\frac{3}{2}-\alpha} \mathbb{E} \left[\left| \int_{s \wedge T_R \wedge \tau_M}^{t \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right|^2 \right] ds. \end{aligned}$$

Using the same techniques as in the estimation of I_2 we have

$$\begin{aligned} \mathbb{E} \left[\left| \int_{s \wedge T_R \wedge \tau_M}^{t \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right|^2 \right] &\leq \mathbb{E} \left[\int_s^t |\Delta(\sigma_W)(u \wedge T_R \wedge \tau_M)|^2 du \right] \\ &\leq C \mathbb{E} \left[\int_s^t \varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,u}^2) du \right]. \end{aligned}$$

Then, it follows that

$$\mathbb{E} [|\tilde{B}_2(t)|^2] \leq C \int_0^t (t-s)^{-\frac{3}{2}-\alpha} \mathbb{E} \left[\int_s^t \varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,u}^2) du \right] ds.$$

Consequently

$$\mathbb{E}[\|B_2\|_{\alpha,t}^2] \leq C \int_0^t \varphi(s, t) \mathbb{E} [\varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2)] ds.$$

Step 4: Combining all estimates, leads to

$$\begin{aligned} &\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2] \\ &\leq C_{M,R} \int_0^t \varphi(s, t) \mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 + \varrho(\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2)] ds. \end{aligned}$$

Since ϱ is concave, Jensen's inequality gives

$$\begin{aligned} &\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2] \\ &\leq C_{M,R} \int_0^t \varphi(s, t) \left[\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2] + \varrho(\mathbb{E} [\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2]) \right] ds. \end{aligned}$$

This concludes the proof. \square

Theorem 7 (Pathwise uniqueness). *Let $1 - H < \alpha < \min(\beta, 1/2)$. Then, under hypotheses (H.1) and (H.2), the pathwise uniqueness property holds for Equation (1.1).*

Proof. It is simple to see that the function $\tilde{\varrho}(u) = u + \varrho(u)$ is a concave increasing function from \mathbb{R}_+ to \mathbb{R}_+ such that $\tilde{\varrho}(0) = 0$ and $\tilde{\varrho}(u) > 0$ for $u > 0$. On the other hand, we have $\varrho(u) \geq \varrho(1)u$ for $0 \leq u \leq 1$. Then

$$\int_{0+} \frac{du}{\tilde{\varrho}^q(u^{1/q})} \geq \left(\frac{\varrho(1)}{1 + \varrho(1)} \right)^q \int_{0+} \frac{du}{\varrho^q(u^{1/q})} = \infty.$$

Therefore, the condition (2.1) is satisfied for the function $\tilde{\varrho}$. Consequently, we can apply Lemma 14 in the Appendix to the inequality (3.3) to obtain

$$\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2 = 0, \text{ a.s.}$$

This implies $X(t) = Y(t)$ a.s. for all $t < T_R \wedge \tau_M$. By letting $M \rightarrow \infty$ we get, by Lemma 5, $X(t) = Y(t)$ a.s. for all $t < T_R$. Using the fact that the random variable $\|B^H\|_{1-\alpha,\infty,t}$ has moments of all orders, see Lemma 7.5 in Nualart and Rascanu [16], it is not difficult that almost surely $T_R = T$ for R large enough. This concludes the proof. \square

4 Euler Approximation scheme

In this section, we apply the Euler approximation procedure in order to obtain a weak solution of Equation (1.1). Under the condition that pathwise uniqueness holds for Equation (1.1) we prove that the Euler approximation converges to a process which is a strong solution of the SDE (1.1), see Theorem 9 below.

Let $0 = t_0^n < t_1^n < \dots < t_i^n < \dots < t_n^n = T$ be a sequence of partitions of $[0, T]$ such that

$$\sup_{0 \leq i \leq n-1} |t_{i+1}^n - t_i^n| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We define Euler's approximations as the process X^n , $n \in \mathbb{N}$, satisfying

$$\begin{aligned} X^n(t) &= x_0 + \int_0^t b(k_n(s), X(k_n(s))) ds + \int_0^t \sigma_W(k_n(s), X(k_n(s))) dW(s) \\ &\quad + \int_0^t \sigma_H(k_n(s), X(k_n(s))) dB^H(s), \end{aligned} \tag{4.1}$$

where $k_n(t) := t_i^n$ if $t \in [t_i^n, t_{i+1}^n)$ and $t \in [0, T]$. For every positive constant R we define the family of stochastic processes by

$$X_R^n(t) := X^n(t \wedge T_R), \quad t \in [0, T].$$

Then it is easy to see that the process X_R^n satisfies, a.s., the following

$$\begin{aligned} X_R^n(t) &= x_0 + \int_0^{t \wedge T_R} b(k_n(s), X^n(k_n(s))) ds + \int_0^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) dW(s) \\ &\quad + \int_0^{t \wedge T_R} \sigma_H(k_n(s), X^n(k_n(s))) dB^H(s). \end{aligned}$$

We obtain for any integer $N \geq 1$

Lemma 8. *Suppose that Assumptions (H.1) and (H.2) hold. Then, for all $n \in \mathbb{N}$, $N \in \mathbb{N}^*$ and $R > 0$, there exists a positive constant $C_{N,R}$ such that*

$$\sup_{t \in [0, T]} \mathbb{E} [\|X_R^n\|_{\alpha, t}^{2N}] \leq C_{N,R}. \quad (4.2)$$

Moreover, we also have for all $s, t \in [0, T]$,

$$\mathbb{E} [|X_R^n(t) - X_R^n(s)|^{2N}] \leq C_{N,R} |t - s|^N. \quad (4.3)$$

Proof. It follows from the convexity of x^{2N} that

$$\begin{aligned} \mathbb{E} [\|X_R^n\|_{\alpha, t}^{2N}] &\leq C_N \left\{ |x_0|^{2N} + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} b(k_n(s), X^n(k_n(s))) ds \right\|_{\alpha, t}^{2N} \right] \right. \\ &\quad + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) dW(s) \right\|_{\alpha, t}^{2N} \right] \\ &\quad \left. + \mathbb{E} \left[\left\| \int_0^{\cdot \wedge T_R} \sigma_H(k_n(s), X^n(k_n(s))) dB^H(s) \right\|_{\alpha, t}^{2N} \right] \right\} \\ &= C_N \left(|x_0|^{2N} + I_1 + I_2 + I_3 \right). \end{aligned}$$

Using the same estimations as in the proof of Lemma 5, we obtain

$$\begin{aligned} I_1 &\leq C_N \left(1 + \int_0^t \mathbb{E} [|X^n(k_n(s) \wedge T_R)|^{2N}] ds \right) \\ &\leq C_N \left(1 + \int_0^t \mathbb{E} [|X_R^n(k_n(s))|^{2N}] ds \right). \end{aligned}$$

$$\begin{aligned} I_2 &\leq C_N \mathbb{E} \left[\left| \int_0^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) dW(s) \right|^{2N} \right] \\ &\quad + C_N \mathbb{E} \left[\left(\int_0^t (t-s)^{-\alpha-1} \left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) dW(u) \right| ds \right)^{2N} \right] \\ &= I_{21} + I_{22}. \end{aligned}$$

For I_{21} , using the linear growth assumption in **(H.1.3)**, Burkholder's and Hölder's inequalities, we obtain

$$\begin{aligned}
I_{21} &\leq C_N \mathbb{E} \left[\int_0^{t \wedge T_R} |\sigma_W(k_n(s), X^n(k_n(s)))|^{2N} ds \right] \\
&\leq C_N \mathbb{E} \left[\int_0^t |\sigma_W(k_n(s) \wedge T_R, X^n(k_n(s) \wedge T_R))|^{2N} ds \right] \\
&\leq C_N \left(1 + \int_0^t \mathbb{E} \left[|X_R^n(k_n(s))|^{2N} \right] ds \right).
\end{aligned}$$

For I_{22} , again the Burkholder and Hölder inequalities give

$$\begin{aligned}
I_{22} &\leq C_N \left(\int_0^t \frac{ds}{(t-s)^{\alpha+\frac{1}{2}}} \right)^{2N-1} \\
&\quad \times \int_0^t (t-s)^{-\alpha-\frac{1}{2}-N} \mathbb{E} \left[\left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) dW(u) \right|^{2N} \right] ds \\
&\leq C_N \int_0^t (t-s)^{-\alpha-\frac{3}{2}} \mathbb{E} \left[\int_{s \wedge T_R}^{t \wedge T_R} |\sigma_W(k_n(s), X^n(k_n(s)))|^{2N} du \right] ds \\
&\leq C_N \int_0^t (t-s)^{-\alpha-\frac{3}{2}} \mathbb{E} \left[\int_s^t |\sigma_W(k_n(u) \wedge T_R, X^n(k_n(u) \wedge T_R))|^{2N} du \right] ds.
\end{aligned}$$

Applying now Fubini's theorem and using the growth assumption in **(H.1.3)**, we obtain

$$I_{22} \leq C_N \int_0^t (t-s)^{-\alpha-\frac{1}{2}} \left(1 + \mathbb{E} \left[|X_R^n(k_n(s))|^{2N} \right] \right) ds.$$

Thus

$$I_2 \leq C_N \left(1 + \int_0^t (t-s)^{-\alpha-\frac{1}{2}} \mathbb{E} \left[|X_R^n(k_n(s))|^{2N} \right] ds \right).$$

Let us remark that

$$\begin{aligned}
&\int_0^{t \wedge T_R} \sigma_H(k_n(s), X^n(k_n(s))) dB^H(s) \\
&= \int_0^t \sigma_H(k_n(s) \wedge T_R, X^n(k_n(s) \wedge T_R)) dB^H(s \wedge T_R) \\
&= \int_0^t \sigma_H(k_n(s) \wedge T_R, X_R^n(k_n(s))) dB^H(s \wedge T_R)
\end{aligned} \tag{4.4}$$

Using (4.4) and Proposition 12 (jj) in the Appendix we obtain

$$I_3 \leq C_N R^{2N} \left(\int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \mathbb{E} [\|X_R^n(k_n(\cdot))\|_{\alpha,s}]) ds \right)^{2N}.$$

By Hölder's inequality we have

$$I_3 \leq C_N R^{2N} \int_0^t \varphi(s, t) (1 + \mathbb{E} [\|X_R^n(k_n(\cdot))\|_{\alpha, s}^{2N}]) ds.$$

Putting all the estimates obtained for I_1 , I_2 and I_3 together, we obtain

$$\mathbb{E} [\|X_R^n\|_{\alpha, t}^{2N}] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(s, t) \mathbb{E} [\|X_R^n(k_n(\cdot))\|_{\alpha, s}^{2N}] ds. \quad (4.5)$$

Therefore, since the right hand side of Equation (4.5) is an increasing function of t , we have

$$\sup_{0 \leq s \leq t} \mathbb{E} [\|X_R^n\|_{\alpha, s}^{2N}] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(s, t) \mathbb{E} \left[\sup_{0 \leq u \leq s} \|X_R^n\|_{\alpha, u}^{2N} \right] ds.$$

As a consequence, by the Gronwall type lemma (cf. Lemma 7.6 in [16]), we deduce the first estimate (4.2) of the lemma. Let us now prove the second estimate (4.3). We have

$$\begin{aligned} & X_R^n(t) - X_R^n(s) \\ &= \int_{s \wedge T_R}^{t \wedge T_R} b(k_n(r), X^n(k_n(r))) dr + \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(r), X^n(k_n(r))) dW(r) \\ & \quad + \int_{s \wedge T_R}^{t \wedge T_R} \sigma_H(k_n(r), X^n(k_n(r))) dB^H(r). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E} [|X_R^n(t) - X_R^n(s)|^{2N}] &\leq C_N \left\{ \mathbb{E} \left[\left| \int_{s \wedge T_R}^{t \wedge T_R} b(k_n(r), X^n(k_n(r))) dr \right|^{2N} \right] \right. \\ & \quad + \mathbb{E} \left[\left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(r), X^n(k_n(r))) dW(r) \right|^{2N} \right] \\ & \quad \left. + \mathbb{E} \left[\left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_H(k_n(r), X^n(k_n(r))) dB^H(r) \right|^{2N} \right] \right\} \\ &= C_N (J_1 + J_2 + J_3). \end{aligned}$$

Applying Hölder's inequality, the growth assumption **(H.1.1)** and (4.2), we have

$$\begin{aligned} J_1 &\leq \mathbb{E} \left[\left(\int_s^t |b(k_n(r) \wedge T_R, X^n(k_n(r) \wedge T_R))| dr \right)^{2N} \right] \\ &\leq C_N (t - s)^{2N-1} \int_s^t \mathbb{E} [|b(k_n(r) \wedge T_R, X_R^n(k_n(r)))|^{2N}] dr \\ &\leq C_N (t - s)^{2N}. \end{aligned}$$

By the Hölder and Burkholder inequalities and using (4.2), we obtain

$$\begin{aligned} J_2 &\leq C_N (t - s)^{N-1} \mathbb{E} \left[\int_{s \wedge T_R}^{t \wedge T_R} |\sigma_W(k_n(r), X^n(k_n(r)))|^{2N} dr \right] \\ &\leq C_N (t - s)^{N-1} \mathbb{E} \left[\int_s^t |\sigma_W(k_n(r) \wedge T_R, X_R^n(k_n(r)))|^{2N} dr \right] \\ &\leq C_N (t - s)^N. \end{aligned}$$

Let us note that we obtain from (2.3) and the Hölder inequality

$$\left| \int_s^t f(u) dB^H(u) \right|^{2N} \leq C_N R^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \int_s^t \frac{\|f(r)\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr.$$

Combining this estimate and (4.4) we obtain

$$J_3 \leq C_N R^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \mathbb{E} \left[\int_s^t \frac{\|\sigma_H(k_n(r) \wedge T_R, X_R^n(k_n(r)))\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr \right].$$

Using the Hölder inequality, assumption **(H.2)** and (4.2), we arrive at

$$\begin{aligned} J_3 &\leq C_N R^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \mathbb{E} \left[\int_s^t \frac{1 + \|X_R^n(k_n(r))\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr \right] \\ &\leq C_N (t-s)^N. \end{aligned}$$

All these estimates allow us to obtain

$$\mathbb{E} \left[|X_R^n(t) - X_R^n(s)|^{2N} \right] \leq C_{N,R} |t-s|^N.$$

The proof of Lemma 8 is then completed. \square

Now we are able to give the convergence result.

Theorem 9. *Assume that σ_W and b are continuous satisfying the linear growth condition. Suppose moreover that σ_H satisfies the assumption **(H.2)** and that for Equation (1.1) the pathwise uniqueness holds. Then Euler's approximations $X^n(t)$ converge to a process $X(t)$ in probability, uniformly in t in $[0, T]$. Furthermore $X(t)$ is the unique strong solution of Equation (1.1).*

Proof. Fix $\eta < 1/2$. We have from (4.3) in Lemma 8 that X_R^n is weakly relatively compact in C_0^η for every R . We want to deduce from this the weak compactness in C_0^η of X^n . Clearly it suffices to show that

$$\limsup_{R \rightarrow \infty} P[T_R \leq T] = 0.$$

This is a consequence of that fact that the random variable $\|B^H\|_{1-\alpha, \infty, t}$ has moments of all orders (see Lemma 7.5 in [16]). We now take two subsequences X^l, X^m of the Euler's approximations X^n . Then obviously (X^l, X^m) is a tight family of processes in $C_0^\eta \times C_0^\eta$. By Skorokhod's embedding theorem there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a sequence $(\tilde{X}^{l,n}, \tilde{X}^{m,n}, \tilde{B}^n, \tilde{W}^n)$ with values in C_0^η such that

1. The law of $(\tilde{X}^{l,n}, \tilde{X}^{m,n}, \tilde{B}^n, \tilde{W}^n)$ and (X^l, X^m, B^H, W) coincide for every $n \in \mathbb{N}$.
2. There exist a subsequence $(\tilde{X}^{l(j)}, \tilde{X}^{m(j)}, \tilde{B}^{n(j)}, \tilde{W}^{n(j)})$ converging in C_0^η to $(\hat{X}, \hat{Y}, \hat{B}, \hat{W})$ uniformly in t , \tilde{P} a.s., that is

$$\lim_{j \rightarrow \infty} \left(\|\tilde{X}^{m(j)} - \hat{X}\|_\eta + \|\tilde{X}^{l(j)} - \hat{Y}\|_\eta + \|\tilde{B}^{n(j)} - \hat{B}\|_\eta + \|\tilde{W}^{n(j)} - \hat{W}\|_\eta \right) = 0.$$

We obtain from Lemma 3.1 in Gyöngy and Krylov [9] and the convergence of integrals with respect to fBms (5.7) in Guerra and Nualart [8] that

$$\begin{aligned}\lim_{j \rightarrow \infty} \int_0^t b(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) ds &= \int_0^t b(s, \hat{X}(s)) ds \\ \lim_{j \rightarrow \infty} \int_0^t \sigma_W(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) d\tilde{W}^{n(j)}(s) &= \int_0^t \sigma_W(s, \hat{X}(s)) d\hat{W}(s) \\ \lim_{j \rightarrow \infty} \int_0^t \sigma_H(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) d\tilde{B}^{n(j)}(s) &= \int_0^t \sigma_H(s, \hat{X}(s)) d\hat{B}(s),\end{aligned}$$

and

$$\begin{aligned}\lim_{j \rightarrow \infty} \int_0^t b(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) ds &= \int_0^t b(s, \hat{Y}(s)) ds \\ \lim_{j \rightarrow \infty} \int_0^t \sigma_W(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) d\tilde{W}^{n(j)}(s) &= \int_0^t \sigma_W(s, \hat{Y}(s)) d\hat{W}(s) \\ \lim_{j \rightarrow \infty} \int_0^t \sigma_H(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) d\tilde{B}^{n(j)}(s) &= \int_0^t \sigma_H(s, \hat{Y}(s)) d\hat{B}(s),\end{aligned}$$

in probability, and uniformly in $t \in [0, T]$. Therefore, the processes \hat{X}, \hat{Y} satisfy the same SDE (1.1), on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, with the driving noises \hat{W}, \hat{B} and the initial condition x_0 on the time interval $[0, \hat{T}_R]$ with

$$\hat{T}_R := \inf \{t \geq 0, \|\hat{B}\|_{1-\alpha, \infty, t} \geq R\} \wedge T, \quad R > 0.$$

Again, as above, we have a.s. $\hat{T}_R = T$ for all R large enough. So that \hat{X}, \hat{Y} satisfy the same SDE (1.1), on $[0, T]$. Then by pathwise uniqueness, we conclude that $\hat{X}(t) = \hat{Y}(t)$ for all $t \in [0, T]$ \tilde{P} a.s.. Hence, by applying Lemma 13 in the Appendix we obtain the convergence of Euler's approximations $X^n(t)$ to a process $X(t)$ in probability, uniformly in t in $[0, T]$. Therefore, $\{X(t), t \in [0, T]\}$ satisfy Equation (1.1). \square

As a consequence we obtain the following existence result.

Theorem 10. *Assume that b, σ_W and σ_H satisfy the hypotheses (H.1) – (H.2). If $1 - H < \alpha < \min(\beta/2, 1)$, then the Equation (1.1) has a unique strong solution.*

Appendix

In this appendix, we recall some results which play a great role in this work. We also show a technical lemma that have been used in the proof of pathwise uniqueness. We begin with some a priori estimates from the paper of Nualart and Rascanu [16].

Proposition 11. *We have*

$$\begin{aligned}(i) \quad & \left\| \int_0^\cdot f(s) ds \right\|_{\alpha, t} \leq C \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds. \\ (ii) \quad & \left\| \int_0^\cdot f(s) dB^H(s) \right\|_{\alpha, t} \leq C \|B^H\|_{1-\alpha, \infty, t} \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \|f\|_{\alpha, s} ds.\end{aligned}$$

Moreover, under the linear growth assumption, we have from Nualart and Rascanu [16], the following

Proposition 12. *Assume (H.1) and (H.2). The following estimates hold*

$$(j) \left\| \int_0^\cdot b(s, f(s)) ds \right\|_{\alpha, t} \leq C \left(\int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + 1 \right)$$

$$(jj) \left\| \int_0^\cdot \sigma_H(s, f(s)) dB^H(s) \right\|_{\alpha, t} \leq C \|B^H\|_{1-\alpha, \infty, t} \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \|f\|_{\alpha, s}) ds$$

We recall the following characterization of the convergence in probability in term of weak convergence, see Gyöngy and Krylov [9].

Lemma 13. *Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random elements in a Polish space (\mathcal{E}, d) equipped with the Borel σ -algebra. Then $(Z_n)_{n \in \mathbb{N}}$ converges in probability to an \mathcal{E} -valued random element if and only if for every pair of subsequences $(Z_m)_{m \in \mathbb{N}}$ and $(Z_k)_{k \in \mathbb{N}}$ there exists a subsequence $(Z_{m(p)}, Z_{k(p)})_{p \in \mathbb{N}}$ converging weakly to a random element v supported on the diagonal $\{(x, y) \in \mathcal{E} \times \mathcal{E} : x = y\}$.*

Finally, let us give a version of the Bihari's lemma.

Lemma 14. *Let $1/2 < \alpha < 1$ and $c \geq 0$ be fixed and $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous function such that*

$$f(t) \leq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} \varrho(f(s)) ds.$$

where ϱ is a concave increasing function from \mathbb{R}_+ to \mathbb{R}_+ such that $\varrho(0) = 0$, $\varrho(u) > 0$ for $u > 0$ and satisfying (2.1) for some $q > 1$. Then for any $1 < p < 2$ such that $\alpha < 1/p$ and $q > 1$ with $1/p + 1/q = 1$ we have

$$f(t) \leq \left[F^{-1} \left(F(2^{q-1}a^q) + 2^{q-1}b^q C_{\alpha, p}^{q/p} t^{q((1/p)-\alpha)+1} \right) \right]^{1/q},$$

for all $t \in [0, T]$ such that

$$F(2^{q-1}a^q) + 2^{q-1}b^q C_{\alpha, p}^{q/p} t^{q((1/p)-\alpha)+1} \in \text{Dom}(F^{-1}),$$

where

$$F(x) = \int_1^x \frac{du}{\varrho^q(u^{1/q})}, \quad \text{for } x \geq 0,$$

and F^{-1} is the inverse function of F . In particular, if moreover, $a = 0$ then $f(t) = 0$ for all $0 < t < T$.

Proof. Let $1 < p < 2$ such that $\alpha < 1/p$. Using the Hölder inequality we obtain

$$f(t) \leq a + bt^\alpha \left(\int_0^t (t-s)^{-p\alpha} s^{-p\alpha} ds \right)^{1/p} \left(\int_0^t \varrho^q(f(s)) ds \right)^{1/q}$$

For the first integral, using $s = tu$, we have the estimate

$$\int_0^t (t-s)^{-p\alpha} s^{-p\alpha} ds = t^{1-2p\alpha} \int_0^1 (1-u)^{-p\alpha} u^{-p\alpha} du = C_{\alpha, p} t^{1-2p\alpha}$$

where $C_{\alpha,p} = B(1 - p\alpha, 1 - p\alpha)$ is the beta function. It follows that

$$f(t) \leq a + b C_{\alpha,p}^{1/p} t^{(1/p)-\alpha} \left(\int_0^t \varrho^q(f(s)) ds \right)^{1/q}.$$

This yields

$$f^q(t) \leq 2^{q-1} a^q + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)} \int_0^t \varrho^q(f(s)) ds.$$

Then it follows from Bihari's Lemma, see [4], that

$$f(t) \leq \left[F^{-1} \left(F(2^{q-1} a^q) + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)+1} \right) \right]^{1/q},$$

for all such $t \in [0, T]$ such that

$$F(2^{q-1} a^q) + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)+1} \in \text{Dom}(F^{-1}).$$

Now, it is simple to see from (2.1) that if $a = 0$ then $f(t) = 0$ for $t \in [0, T]$. □

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